An important and nice result in CDMA asymptotic analysis

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Remainder: CDMA channel model

\[ y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t) \]

where:
- \( K \): Number of users.
- \( s_k \): Unity energy signature waveform of the \( k \)th user.
- \( b_k \): Data symbol of the \( k \)th user.
- \( A_k \in \{1, -1\} \): Received amplitude of the \( k \)th user.
- \( n(t) \): Additive Gaussian noise.
Uncoded bit error rate is the main performance measure of multiuser detectors.

Multiuser Efficiency is an alternative way to characterize the multiuser bit error rate.

OAME is equal to:

\[ \eta_k = \min_{v \in \{-1,0,1\}^K, v_k = 1} \left( \frac{1}{A_k^2} v^T A R A v \right) \]
Optimum asymptotic Multiuser Efficiency

\[ \eta_k = \min_{v \in \{-1,0,1\}^K, v_k = 1} \left( \frac{1}{A_k^2} v^T A R A v \right) \]

where:

- \( A = \text{diag}(A_1, \ldots, A_K) \)
- \( R \) is the normalized crosscorrelation matrix with entries:

\[ \rho_{ij} = \langle s_i, s_j \rangle \]
Multiuser efficiency for linear detectors

- Single-user matched filter: 
  \[ \eta^c \rightarrow 0 \]
- Decorrelator: 
  \[ \eta^d \rightarrow [1 - \beta]^+ \]
- Linear MMSE: 
  \[ \eta^d \rightarrow [1 - \beta]^+ \]

The asymptotic efficiency decreases as \( \beta \uparrow \) and vanishes for \( \beta > 1 \).
New Result on Optimum detectors

Theorem: Assuming:

- $|E[|s_nk|^3]| < \infty$, $n = 1, \ldots, N$, $k = 1, \ldots, K$
- $\theta < \frac{A_k}{A_1} < \gamma$, $k = 2, \ldots, K$, $\theta, \gamma > 0$

In the large-system limit, $N, K \to \infty$, $K/N \to \beta > 0$, the optimum multiuser efficiency for user 1 converges in probability to 1 almost surely.
Proof

- We define:

\[ C_K = \{ v \in \{-1, 0, 1\}, \ v_1 = 1, \ v_j \neq 0, \ j > 1 \} \]

- \[ E_K = \{ v^T A \mathbf{R} A v < 1, \text{ for some } v \in C_K \} \]

- We shall prove:

\[ \lim_{N \to \infty} \Pr[E_K] = 0 \]

and apply Borel-Cantelli Lemma over \( K \).
The key expression is divided into two sums:

\[
\Pr\{E_K\} \leq \sum_{v \in C_K} \Pr[v^T\text{ARA}v < 1]
= \sum_{v \in C_K, w(v) \leq M_0} \Pr[v^T\text{ARA}v < 1] \\
+ \sum_{v \in C_K, w(v) > M_0} \Pr[v^T\text{ARA}v < 1]
\]

where \(w(v)\) is the number of non-zero components and \(M_0 \geq 2\).
First Sum

- Define the random variable:

$$Y_n = \left[ s_{n,1} + \sum_{k=2}^{K} v_K A_K s_{n,K} \right]^2$$

- It can be seen that:

$$\frac{1}{N} \sum_{n=1}^{N} Y_n = v^T ARAv$$
First Sum

- Use of Chernoff bound:

\[ \Pr[v^T A R A v < 1] = \Pr \left[ \frac{1}{N} \sum_{n=1}^{N} Y_n \right] \leq \exp[-NI_v(1)] \]

where \( I_v(1) \) is the large deviation rate.

- Authors find a positive lower bound on \( I_v(1) \) uniform for all \( v \in C_K \) and \( w(v) \leq M_0 \)
Second Sum

Use of Berry-Esseen refinement of the Central Limit Theorem to obtain the following bound:

\[ |E[\exp(rY)] - E[\exp(r\tilde{Y})]| \leq \frac{B_1}{\sqrt{w(v)}} \]

where \( \tilde{Y} \) is a random variable with the same mean as \( Y \) and \( \chi^2 \) is some constant.
Application of Chernoff bound:

\[ \Pr[v^T A R A v < 1] = \Pr \left( \frac{1}{N} \sum_{n=1}^{N} Y_n \right) \leq \exp\left[ -N \left( r - \log E[\exp(r \tilde{Y})] + \frac{B_1}{\sqrt{w(v)}} \right) \right] \]

Setting \( r = -1 \) and bounding \( E[\exp(r \tilde{Y})] \)

\[ \Pr[v^T A R A v < 1] \leq \exp \left( -N \left( \frac{1}{\sqrt{1 + 2r \theta^2 w(v)}} + \frac{B_1}{\sqrt{w(v)}} \right) \right) \]

\[ \leq \exp \left( -N \left( \frac{1}{\sqrt{1 + 2 \theta^2 w(v)}} + \frac{B_1}{\sqrt{w(v)}} \right) \right) \]
The system load $\beta$ used to bound the second sum:

$$\sum_{v \in C_K, w(v) > M_0} \Pr[v^T A R A v < 1]$$

$$\leq \sum_{v \in C_K, w(v) > M_0} \exp \left[ -N \left\{ -1 + \log \left( \frac{\sqrt{w(v)}}{B_2} \right) \right\} \right]$$

$$< |C_K| \exp \left[ -N \left\{ -1 + \log \left( \frac{\sqrt{M_0}}{B_2} \right) \right\} \right]$$

$$< 3^{\beta N} \exp \left[ -N \left\{ -1 + \log \left( \frac{\sqrt{M_0}}{B_2} \right) \right\} \right]$$

$$= \exp \left[ -N \left\{ -1 - \beta \log 3 + \log \left( \frac{\sqrt{M_0}}{B_2} \right) \right\} \right]$$
Remarks

- For \( M_0 \) large enough, the second sum also goes to zero exponentially in \( N \).

- Hence, combining both sums for an appropriate value of \( M_0 \) yields the result.

- The uniform lower bound was used in the first sum and the upper bound condition in the second one.
The proof of convergence is based on:

- Chernoff large deviation bounds.
- Refinement of the Central Limit Theorem.

The theorem holds for non-disparities on the amplitudes. If the amplitudes are not subjected to restriction, the OEME does not converge to 1.
Conclusions

- Qualitatively, it shows that the asymptotic behavior of a CDMA multiple access channel for binary data transmission is equivalent to that of the single user system.

- In large-system conditions, the nearest neighbor to each of transmitted signals is at the same distance as in the absence of interferers.